Full Counting Statistics of a charge shuttle

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We study the charge transfer in a small grain oscillating between two leads. Coulomb blockade restricts the charge fluctuations in such a way that only zero or one additional electrons can sit on the grain. The system thus acts as a charge shuttle. We obtain the full counting statistics of charge transfer and discuss its behavior. For large oscillation amplitude the probability of transferring $\tilde{n}$ electrons per cycle is strongly peaked around one. The peak is asymmetric since its form is controlled by different parameters for $\tilde{n} > 1$ and $\tilde{n} < 1$. Under certain conditions the systems behaves as if the effective charge is $1/2$ of the elementary one. Knowledge of the counting statistics gives a new insight on the mechanism of charge transfer.

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I. INTRODUCTION

A few years ago Gorelik *et al.* showed that a small conducting grain can undergo a mechanical instability if it is trapped in a harmonic potential between two leads kept at a constant voltage bias (see Fig. 1).[1] When the central grain is charged, the electrostatic force induced by the leads pushes it towards one of the electrodes, increasing the probability that the excess charge be discharged. Since the resistance depends exponentially on the distance, even small oscillations can largely amplify the probability of transmission. Excitation of this nanomechanical system at one of its resonating frequencies can be generated by the stochastic tunnelling of electrons from the leads. Since the charge state of the grain is correlated with its position, under certain conditions, the energy accumulated in the mechanical systems increases indefinitely, leading to an instability. The energy pumped depends on the oscillation amplitude up to a maximum value determined by the number of charges that can accumulate in the grain at each cycle. After that point there is no additional gain in increasing the amplitude and the grain stabilizes at a fixed oscillation amplitude for which the energy pumped exactly balances the energy dissipated. This scenario has been investigated both in the incoherent[1,2,3] and in the quantum case.[4,5,6,7,8,9]. There are strong indications that Parks *et al.* have observed this phenomenon in C_{60} molecules oscillating between two leads.[10]

An other possibility to drive the oscillations is to use an external alternate electric field acting on a a cantilever. The amplitude can thus be tuned independently of the source/drain voltage bias, at least in principle. This case has been experimentally realized by Erbe *et al.* who observed a current of 0.11 electrons per cycle at low temperature induced by the oscillation of the central grain. When the leads and the grain are superconducting the existence of phase coherent transport as been proposed.[12,13,14]

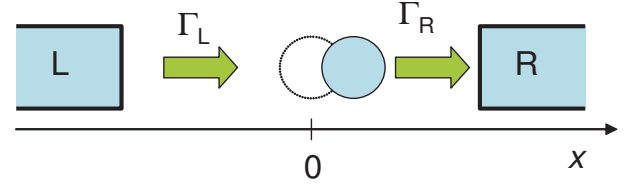Many papers studied theoretically the conditions for

FIG. 1: Simple schematization of the system. A small grain oscillating in an harmonic potential between two leads kept at a constant voltage bias.

the realization of the instability or considered the dependence of the current on the external parameters. Only few papers investigated instead current fluctuations. Weiss and Zwerger[15] calculated the average number of electrons transmitted and its fluctuation during *a single* cycle of a shuttle driven at a given frequency and amplitude. This quantity differs from the noise actually measured since typical measurement times are much longer than one period of oscillation. Correlations of charge fluctuations on different cycles are then important, as we discuss in the following. Other authors considered the finite frequency noise in superconducting shuttles[14] or the telegraph noise induced by the switching between two mechanical modes in a two-oscillating-grains device.[16] A related problem is the fluctuation of the acoustoelectric current carried by surface acoustic waves propagating along a ballistic quantum channel.[17]

The full counting statistics (FCS) of charge transfer in a shuttle has not been considered so far. Recently, powerful techniques have been developed to calculate the probability than n electrons are transferred during a measurement time t_o in electronic devices.[18,19,20,21] The FCS contains much more information on the dynamics of the charge transfer than the current or the noise alone. This will be particularly clear in this problem since few electrons are involved in the tunnelling, and the probability that $\tilde{n}$ electrons per cycle are transmitted is actually a fundamental quantity. For a well developed shuttling

regime the noise to current ratio is expected to be small, since the number of electrons shuttled at every cycle is determined by the Coulomb blockade conditions and thus it does not fluctuate as it happens in a purely stochastic tunnelling. This implies that the probability distribution has a small width, but its actual shape still depends on the physical parameters of the junction, like voltage bias, tunnelling probability, or oscillation amplitude of the shuttle. The importance of studying theoretically the FCS is thus twofold: first it is, at least in principle, a measurable quantity and secondly, its knowledge allows to infer detailed information on the mechanism of charge transfer.

The paper is organized as follows. In Sec. II the technique for the calculation is developed and the equations for the numerical approach are obtained. In Sec. III the FCS is calculated analytically for small and large oscillations. In Sec. IV the general (numerical) results are discussed and compared with the analytical ones. Section V gives our conclusions.

II. FULL COUNTING STATISTICS FOR AN OSCILLATING GRAIN

Our aim is to calculate the FCS of charge transfer in an oscillating grain between two leads. We assume that the charge transfer is described by the "orthodox theory" of Coulomb blockade.[22,23] In this regime the tunnelling is incoherent and the dynamics is governed by a standard master equation. Within these assumptions Bagrets and Nazarov have developed an elegant and efficient technique to derive the FCS for the static case.[20] We will use their method generalized to a moving grain.

Since we are interested to a single grain structure, the state of the system is completely determined by the probability p_k of having k additional electrons in the island. For simplicity we consider the case where the voltage biases guarantee that only the two states, $k = 0, 1$, are available, and that only two events are possible: either one electron jumps on the island from the left lead [with transition rate $\Gamma_L(t)$] or one electron on the island (if present) jumps to the right lead [with transition rate $\Gamma_R(t)$]. Within these assumptions the time evolution of the probability is given by a master equation. We write it following the notation of Ref. 20:

$$\frac{\partial}{\partial t}|p(t)\rangle = -\hat{L}(t)|p(t)\rangle \qquad (1)$$

where the (classical) probability is represented by a state in a vector space: $|p\rangle = \{p_0, p_1\}$, and $\hat{L}$ is the matrix

$$\hat{L}(t) = \begin{pmatrix} \Gamma_L(t) & -\Gamma_R(t) \\ -\Gamma_L(t) & \Gamma_R(t) \end{pmatrix}. \qquad (2)$$

In Ref. 20 it is shown how the FCS can be obtained by calculating the time evolution of the probability with a modified operator $\hat{L}(t)$. The central quantity is $P_{t_o}(n)$,

the probability that n electrons have been transmitted during a measurement time t_o. This quantity is independent from the initial condition in the limit of large t_o. From the technical point of view it is easier to calculate the generating function $S_{t_o}(\chi)$:

$$e^{-S_{t_o}(\chi)} = \sum_{n=0}^{\infty} P_{t_o}(n)e^{in\chi}. \qquad (3)$$

From S one can easily obtain all cumulants: $\overline{n} = \partial(-S)/\partial(i\chi)|_{\chi=0}$, $\overline{(n-\overline{n})^2} = \partial^2(-S)/\partial(i\chi)^2|_{\chi=0}$, etc. Let us count electrons crossing, for instance, the left junction. According to the prescriptions of Ref. 20 the modified operator $\hat{L}_\chi(t)$ is obtained from Eq. (2) by multiplying the lower off-diagonal matrix element by the factor $e^{i\chi}$. This factor keeps track of the electrons that cross the left junction during the time evolution. The generating function $S_{t_o}(\chi)$ is then simply given by the formal integration of the modified time-evolution equation:

$$e^{-S_{t_o}(\chi)} = \langle q|\text{Texp}\left\{-\int_0^t \hat{L}_\chi(t')dt'\right\}|p(0)\rangle, \qquad (4)$$

where $|p(0)\rangle$ is the probability at time $t = 0$, $|q\rangle \equiv \{1,1\}$, and Texp is the time ordered exponential. The derivation of this equality was done in Ref. 20 for the static case where Γ's do not depend on time. Following the steps of their proof it is not difficult to verify that Eq. (4) holds also in the dynamical case of interest here. The main difference is that for the static case (and the zero frequency noise) one can restrict to the study of the eigenvalues of $\hat{L}$, since the time ordering becomes immaterial when $\hat{L}$ does not depend on time. In our case instead time-ordered exponential must be evaluated explicitly.

A. Specific expressions for the shuttle

Let us now consider explicitly the time dependence of the tunnelling rate. With good accuracy one can assume that the dependence of $\Gamma_{L/R}$ on the position of the grain is exponential

$$\Gamma_{L/R} = \Gamma^0 \exp\{\mp x/\lambda\} \qquad (5)$$

(we assume $\Gamma_L = \Gamma_R = \Gamma^0$ for $x = 0$). Here x is the shift of the grain from the equilibrium position and λ is the tunnelling length (see also Fig. 1). We will consider the case of sinusoidal oscillations of the grain. This can be driven by an external device like in the experiment of Ref. 11, or it can be induced by the voltage bias between the left and right leads.[1,10] In both cases

$$\Gamma_{L/R}(t) = \Gamma^0 \exp\{\mp a\sin(\omega t)\}, \qquad (6)$$

where $a = x_{max}/\lambda$ is the dimensionless ratio of the oscillation amplitude to the tunnelling length and ω is the frequency of oscillation. It is also convenient to rescale

the time by ω^{-1} and define $\phi = \omega t$. With this substitution the problem is fully characterized by the two dimensionless parameters: a and $\Gamma = \Gamma^0/\omega$. From the physical point of view, Γ gives the probability that an electron in the static junction with $x = 0$ can tunnel on or off the grain in the time $1/\omega$. We will see that the dependence on a of the FCS will be qualitatively different if Γ is smaller or larger than one.

The interesting physical quantity is the FCS for a long measurement time t_o. We choose t_o to be a multiple of the period: $t_o = 2\pi N/\omega$, with N integer. The FCS of charge transfer during N periods is then given by

$$e^{-S_N(\chi)} = \langle q|\hat{A}^N|p(0)\rangle \tag{7}$$

where

$$\hat{A} = \text{Texp}\left\{-\int_0^{2\pi} \hat{L}_\chi(\phi')d\phi'\right\} \tag{8}$$

and

$$\hat{L}_\chi(\phi) = \begin{pmatrix} \Gamma e^{-a\sin\phi} & -\Gamma e^{a\sin\phi} \\ -\Gamma e^{-a\sin\phi}e^{i\chi} & \Gamma e^{a\sin\phi} \end{pmatrix}. \tag{9}$$

In the case of interest of large N the FCS is given by the eigenvalue $\lambda_M(\chi)$ of $\hat{A}$ that has the maximum absolute value: $S_N(\chi) = -N\ln[\lambda_M(\chi)]$. From $S_N(\chi)$ one can calculate directly the particle current and noise reduced to a period: $I = \overline{\tilde{n}} = \overline{n}/N$ and $P = 2\overline{(n-\overline{n})^2}/N$.

To obtain the probability of having transferred n electrons during N periods it suffices to invert Eq. (3):

$$P_N(n) = \int_{-\pi}^{+\pi} \frac{d\chi}{2\pi} e^{-S_N(\chi)-i\chi n}. \tag{10}$$

For large N the saddle point approximation gives a very accurate estimate of this integral:

$$\ln[P_N(n)]/N = \ln[\lambda_M(\chi_0)] - i\chi_0\tilde{n} \tag{11}$$

where $\tilde{n} = n/N$ is the number of electrons transferred per cycle and χ_0 satisfies the equation:

$$\frac{1}{\lambda_M(\chi_0)} \frac{d\lambda_M}{d\chi}\bigg|_{\chi=\chi_0} = i\tilde{n}. \tag{12}$$

We find that Eq. (12) is solved by χ_0 pure imaginary.

The problem is now reduced to the evaluation of the time-ordered product that enters the definition of $\hat{A}$. This can be done numerically by integrating the system of differential equations

$$\frac{\partial}{\partial\phi}|p(\phi)\rangle = -\hat{L}_\chi(\phi)|p(\phi)\rangle \tag{13}$$

with the two initial conditions $|p^{(1)}(\phi = 0)\rangle = \{1, 0\}$ and $|p^{(2)}(\phi = 0)\rangle = \{0, 1\}$. One can readily verify that the matrix with columns $|p^{(1)}(\phi = 2\pi)\rangle$ and $|p^{(2)}(\phi = 2\pi)\rangle$ coincides with $\hat{A}$. In the case of interest the numerical task is not hard, nevertheless discussion of tractable analytical limits greatly enhances the understanding of the results. We thus discuss in the next section the small and large amplitude limits before presenting the numerical results for the general case in Sec. IV.

III. ANALYTICAL LIMITS FOR SMALL AND LARGE AMPLITUDE

A. Static case and fractional charge

For $a = 0$ we have a standard static single electron transistor. Since $\hat{L}_\chi$ does not depend on ϕ, the time-ordered exponential becomes a simple exponential

$$\hat{A} = e^{-2\pi\hat{L}_\chi} \tag{14}$$

and the generating function for large N can be obtained by diagonalization of the $\hat{L}_\chi$ matrix. The smallest eigenvalue in modulus gives $S_N(\chi)$ at the leading order. The generating function reads

$$-S_N(\chi)/N = 2\pi\Gamma\left(e^{i\chi/2} - 1\right), \tag{15}$$

in agreement with the result obtained with a different technique in Ref. 24. The current and the noise are thus: $I = P = \pi\Gamma$ with a Fano factor $F = P/2I$ equal to $1/2$.

Even if Eq. (15) has been derived before its meaning has not been fully discussed and it is worth a short digression. Tunnelling through a single barrier is a Poissonian process. The generating function and the probability in this case is

$$-S_{t_o}(\chi) = \overline{n}\left(e^{i\chi} - 1\right) \quad \text{and} \quad P_{t_o}(n) = e^{-\overline{n}}\frac{\overline{n}^n}{n!}, \tag{16}$$

with $\overline{n}$ the average number of charges transmitted during the time t_o. A general feature of the generating function is the 2π periodicity. It is a manifestation of the discrete nature of the charge and follows directly from the definition (3). It is thus surprising that the generating function (15) is periodic of 4π as if the elementary charge was not one, but $1/2$. This happens only when $\Gamma_L = \Gamma_R$ and for long measurement times ($N \to \infty$), in all other cases $S(\chi)$ is periodic of 2π. For instance if $\Gamma_L \neq \Gamma_R$, even in the large N limit we have the following 2π-periodic generating function

$$\frac{S_N(\chi)}{N(2\pi/\omega)} = \frac{\Gamma_L + \Gamma_R}{2} - \sqrt{\frac{(\Gamma_L - \Gamma_R)^2}{4} + \Gamma_L\Gamma_R e^{i\chi}}. \tag{17}$$

The result (15) for $\Gamma_L = \Gamma_R$ indicates that the charge transfer in our system (Fig. 2 a) is equivalent to that happening in a tunnel junction with charges $1/2$ emitted with probability Γ_0 (Fig. 2 b).

This can be understood with simple arguments. Let us consider a sequence of events in our system. These can be of two types, either tunnelling from the left lead to the grain (type L), or tunnelling from the grain to the right lead (type R). Since the rates for both events are the same (Γ^0), the system has the same probability per unit time to switch to the other state. The statistics of the switching events (i.e. that either R or L occurs, without specifying which one) follows thus a Poissonian
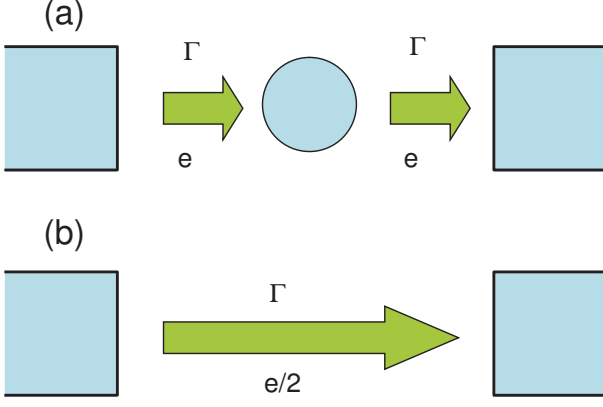
FIG. 2: The two equivalent systems for the statistics of charge transferred in the long measurement time limit: (a) single electrons transistor with equal probability Γ of hopping from the left lead to the grain and from the grain to the right lead (b) a fictitious system of charges $1/2$ that tunnel with probability Γ.

distribution, since all events are independent (rates do not depend on the initial states).

To obtain the statistics of charge transfer from the statistics of switching events it is enough to remember that every two switching events one charge is transmitted. We can thus associate the transmission of a fictitious $1/2$ charge at each switching event. It is clear that the counting statistics of the fictitious charge coincides with that of the true charge, apart from a possible charge $1/2$ mismatch that is irrelevant for long measurement times $(t_o \Gamma^0 \rightarrow \infty)$. Thus we proved that the counting statistics of our system coincides with that predicted for a tunnelling junction of charges $1/2$ transmitted with rate Γ^0: simple classical correlations can induce current fluctuations typical for fractional charges! A similar "fractional" behavior was also found and discussed by Andreev and Mishchenko for charge pump in the Coulomb blockade regime.[25]

One should keep in mind that for any finite measurement time the periodicity of the generating function remains 2π, it is only the leading term for $t_o \rightarrow \infty$ that is 4π periodic. We thus expect that the Fourier series 3 is not uniformly convergent and that for any finite t_o higher moments will definitely depart from the prediction obtained with Eq. (15). This can be verified by calculating the first correction to Eq. (15) (we recall that $2\pi N\Gamma = t_o \Gamma_o$):

$$-\frac{S_N}{2\pi N\Gamma} = \left(e^{i\chi/2} - 1\right) + \frac{1}{2\pi N\Gamma} \log(1 + \cos\chi/2) + \dots, \tag{18}$$

that holds with accuracy $O(e^{-2\pi N\Gamma})$. From Eq. (18) we derive all moments $M_q \equiv \partial^q(-S)/\partial(i\chi)^q$. For q even the

result is

$$M_q = \frac{2\pi N\Gamma}{2^q} \left[1 + \frac{(-1)^{q/2+1}}{2\pi N\Gamma} \frac{4\zeta(q)}{\pi^q}(1 - 2^{-q})(q-1)!\right], \tag{19}$$

while for q odd the correction in Eq. (18) gives a vanishing contribution. For large q and fixed N the second term in Eq. (19) is approximatively $4(q/e\pi)^q/(2\pi N\Gamma)$. This means that for q large enough M_q will depart from the prediction of Eq. (15). This is the way in which the system may reveal the true nature of the elementary charge. Either by a short time measurement of the first moments, or by the long time measurement of higher moments.

To explain while for asymmetric tunnelling rates the periodicity is 2π even for large t_o it is enough to notice that the above discussed mapping cannot be realized when $\Gamma_L \neq \Gamma_R$. It is worth mentioning that for not too large asymmetries the fractional charge remains measurable, as it is clear from the dependence of the Fano factor:[24,26,27]

$$F = \frac{\Gamma_L^2 + \Gamma_R^2}{(\Gamma_L + \Gamma_R)^2} \,. \tag{20}$$

But again one expects that departure from the prediction of Eq. (15) will increase with the order q of the moment, and for any small asymmetry it will become large for q large enough. Generating functions with periodicity induced by smaller fractions of the elementary charge can be obtained with several islands.

B. Large oscillation amplitude

Let us now consider the opposite limit of large oscillation amplitude of the shuttle. In this limit, since for most of the time the ratio Γ_L/Γ_R is either very large or very small, we can assume that (i) for $0 < \phi < \pi$ the quantity Γ_L vanishes identically and (ii) for $\pi < \phi < 2\pi$ the opposite holds: $\Gamma_R = 0$. The approximation becomes exact for $\Gamma \ll 1$, since in that case electrons can tunnel only when the shuttle is near one of the two leads.

Within this approximation $\hat{A}$ can be obtained analytically. As a matter of facts in region (i) $p_0(\phi) + p_1(\phi)$ is conserved, since the matrix element $\hat{L}$ that multiplies $e^{i\chi}$ vanishes. (We recall that $p_k(\phi)$ is the probability that k electrons are present in the grain at time ϕ.) The introduction of the counting field normally breaks the conservation of the probability $p_0(\phi) + p_1(\phi) = const$. Using this conservation and integrating the remaining differential equation, we obtain for region (i)

$$\begin{cases} p_0(\pi) = p_0(0) + (1-\alpha)p_1(0) \\ p_1(\pi) = \alpha p_1(0) \end{cases} \tag{21}$$

where $1 - \alpha$ is the probability of transferring one electron during half cycle given with

$$\alpha = \exp\left\{-\Gamma \int_0^\pi e^{a\sin\phi} d\phi\right\} \,. \tag{22}$$

In region (ii) $p_0(\phi) + e^{-i\chi}p_1(\phi)$ is conserved and we find

$$\begin{cases} p_0(2\pi) = \alpha p_0(\pi) \\ p_1(2\pi) = p_1(\pi) + (1-\alpha)_0^p(\pi)e^{i\chi} \end{cases} \quad . \qquad (23)$$

By composing the evolution in (i) and (ii) we find

$$\hat{A} = \begin{pmatrix} \alpha & \alpha(1-\alpha) \\ (1-\alpha)e^{i\chi} & \alpha + (1-\alpha)^2 e^{i\chi} \end{pmatrix} \qquad (24)$$

The generating function is obtained by diagonalization of Eq. (24):

$$\lambda_M = \alpha + (1-\alpha)\frac{y}{2}\sqrt{(1-\alpha)^2 y^2 + 4\alpha} + (1-\alpha)^2\frac{y^2}{2} , \quad (25)$$

where we introduced the short hand notation $y = e^{i\chi/2}$. Current and noise follows by differentiation:

$$I = \frac{1-\alpha}{1+\alpha}, \qquad P = 4\alpha\frac{1-\alpha}{(1+\alpha)^3} . \qquad (26)$$

For $\Gamma \ll 1$ Eq. (25) is very accurate and holds for $0 \le \alpha \le 1$. It is instructive to study its behavior in the two opposite limits of $\alpha \ll 1$ and $1 - \alpha \ll 1$.

For $\alpha \ll 1$ the probability of transferring the electron during the half cycle is nearly 1. Linearizing Eq. (25) in α we find the generating function of a binomial distribution:

$$e^{-S_N(\chi)} = [2\alpha + (1-2\alpha)e^{i\chi}]^N . \qquad (27)$$

This means that at each cycle one electron is transmitted with probability $1 - 2\alpha$. The cycles are independent: after N cycles the probability of having transmitted n electrons is simply given by the binomial distribution $\binom{N}{n}(1-2\alpha)^n(2\alpha)^{N-n}$. Cycles are independent for $\alpha \to 0$ since at each cycle the system is reset to the stationary state within accuracy α^2, regardless of the initial state. The stationary solution is given by the eigenvector of Eq. (24) with eigenvalue 1 for $\chi = 0$: $|p_{st}\rangle = \{\alpha/(1+\alpha), 1/(1+\alpha)\}$. Calculating the transmission probability for one electron during one cycle with initial condition given by $|p_{st}\rangle$ one obtains, with linear accuracy, the correct result $1 - 2\alpha$ appearing in Eq. (27).

For $\alpha \to 1$ the probability for one electron to tunnel during a cycle is very small. We can thus expand the generating function in the positive quantity $(1-\alpha) \ll 1$. This gives the following surprising result:

$$e^{-S_N(\chi)} = \left[\alpha + (1-\alpha)e^{i\chi/2}\right]^N . \qquad (28)$$

We find again that the periodicity of the generating function has changed. Eq. (28) describes a system of $1/2$ charges that at each cycle have a probability $1 - \alpha$ of being transmitted. The situation is similar to the static case. We can again create a mapping on a fictitious system of charges $1/2$ and say that every time that one electron succeeds in jumping on or off the central island, one

charge $1/2$ is transmitted in the fictitious system. This is possible, since it is extremely unlikely that one electron can perform the full shuttling in one cycle. Thus after many cycles ($N \gg 1$) the counting statistics of these two systems coincide. The cycles are no more independent like in the case for $\alpha \ll 1$, but the problem can be mapped onto an independent tunnelling one. For α intermediate it is more difficult to give a simple interpretation of Eq. (25), since different cycles are correlated in a non trivial way.

In Ref. 15 the current and noise within a similar model have been calculated, but only for a single cycle using the the stationary solution as initial condition. This approach clearly neglects correlations among different cycles. We have seen that this is an excellent approximation for $\alpha \to 0$: Eq. (27) represents N uncorrelated cycles. But it fails completely in the opposite limit of $\alpha \to 1$, where the main contribution to the current fluctuations comes from the cycle-cycle correlations. This can be seen as follows. Starting from the stationary solution for $\alpha \to 1$ (i.e. $\{1/2, 1/2\}$) one can calculate the average number of particles transmitted $\overline{n} = (1-\alpha)/2$ and its fluctuation $\overline{(n-\overline{n})^2} = (1-\alpha)/2$ during a *single* cycle. From Eq. (28) we see that the average current over a large number of cycles is correctly reproduced, but the noise differs by a factor of 2. This difference increases with higher moments. Even if the fluctuation during a single cycle is an interesting physical quantity, the experimentally relevant one is the long time fluctuations.

"Mixing" regions

We expect that Eq. (25) describes pretty well the counting statistics $P_N(n)$ for $n < N$, but it is clear that it fails completely for $n > N$ for which it gives $P_N(n > N) = 0$ identically. As a matter of fact, the approximation does not take into account than more than one electron per cycle can be transmitted. This is an artefact of the assumption that Γ_L and Γ_R are never non-vanishing at the same time.

In order to improve the approximation, but keeping the problem solvable analytically, we need to treat differently the left and right "mixing" regions $\phi \approx 0$, π, 2π. In these regions at lowest order $\Gamma_L \approx \Gamma_R$. We thus divide the time evolutions in 5 steps. For $\phi < \phi_0$, $|\phi - \pi| < \phi_0$, and $2\pi - \phi < \phi_0 \sim 1/a \ll 1$ we calculate the evolution with $\Gamma_L = \Gamma_R = \Gamma^0$. For $\phi_0 < \phi < \pi - \phi_0$ and $\pi + \phi_0 < \phi < 2\pi - \phi_0$ we use instead the previous approximation for regions (i) and (ii). The approximation is summarized in Fig. 3 where the exact and the approximate dependence of $\Gamma_L(\phi)$ is shown.

The contribution to $\hat{A}$ of the three mixing regions depends on ϕ_0 and Γ only through their product $\Gamma\phi_0 \equiv \tau$. The approximation is meaningful only for α small, otherwise the constant approximation for the probabilities in the mixing regions would not be accurate. We thus consider only the small α limit. Keeping linear terms in
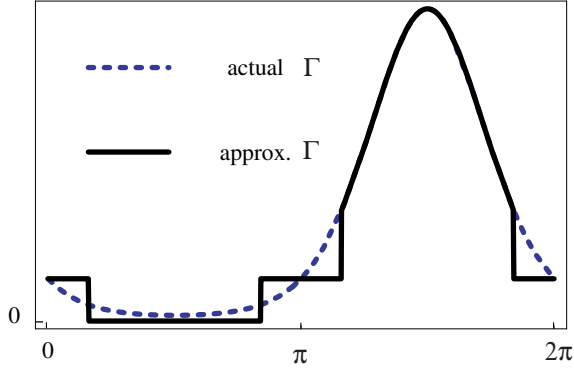
FIG. 3: Exact and approximate dependence on ϕ of Γ_L used to obtain analytically the FCS in the large oscillation limit. $\Gamma_R(\phi)$ has the same form shifted by π in the ϕ axis.

α we have

$$\lambda_M(\chi) = \frac{e^{-4\tau}}{2}\left[(1-2\alpha)(y^2-1)+2y\sinh(4\tau y)\right.$$
$$\left.+\left(1+2\alpha+(1-2\alpha)y^2\right)\cosh(4\tau y)\right] \quad (29)$$

with the same short hand notation $y = e^{i\chi/2}$.

This expression, through $-S_N(\chi)/N = \ln\lambda_M(\chi)$ gives the FCS for large a in different limits. Let us begin with the case $\tau \ll 1$. Expanding Eq. (29) up to second order in τ we obtain

$$e^{-S_N(\chi)} = \lambda_M(\chi)^N = \left[\beta_0 + \beta_1 e^{i\chi} + \beta_2 e^{2i\chi}\right]^N \quad (30)$$

with

$$\begin{cases} \beta_0 = 2\alpha(1-4\tau+8\tau^2) \\ \beta_1 = 1-2\alpha(1-4\tau+4\tau^2)-4\tau^2 \\ \beta_2 = 4\tau^2(1-2\alpha) \end{cases} \quad (31)$$

The interpretation is again simple, Eq. (30) gives a trinomial distribution, at each cycle there is a probability $\beta_{\tilde{n}}$ of transmitting $\tilde{n}$ electrons per cycle. This approximation holds for any $0 < \tilde{n} < 2$ and $\tau \approx \Gamma/a \ll 1$, $a \gg 1$. In this case the probability of transmitting more than two electrons within a cycle is extremely small since $\beta_{\tilde{n}>1} \sim \tau^{2\tilde{n}-2}$. The importance of the parameter τ for $\beta_{\tilde{n}>1}$ proves that to understand the probability of charge transfer for $n > N$ it is crucial to correctly treat the mixing regions where Γ_L and Γ_R are both non-vanishing. This means that in this limit the FCS for $\tilde{n} = n/N > 1$ is determined mainly by the value of τ, while for $0 < \tilde{n} < 1$ is α that controls the FCS.

Using the saddle point approximation, it is possible to obtain explicitly the probability for $\tilde{n} \approx 1$:

$$\frac{\ln[P_N(\tilde{n})]}{N} = \begin{cases} \ln\beta_1 - (1-\tilde{n})\ln\left(\frac{1-\tilde{n}}{\beta_0/\beta_1}\right) & \text{for } \tilde{n} < 1 \\ \ln\beta_1 - (\tilde{n}-1)\ln\left(\frac{\tilde{n}-1}{\beta_2/\beta_1}\right) & \text{for } \tilde{n} > 1. \end{cases} \quad (32)$$

The probability has a sharp maximum at $\tilde{n} = 1$, as expected, and its logarithm decreases approximately linearly on both sides, with slopes controlled by two different parameters. For $\tilde{n} < 1$ the slope is approximately

given by $-\ln(2\alpha)$, while for $\tilde{n} > 1$ it is given by $\ln(4\tau^2)$. Since the parameter α decreases exponentially with a, while τ is only inversely proportional to a, the peak around $\tilde{n} = 1$ is asymmetric with an excess to the left for moderately large a, and with an excess to the right for larger a.

When $\tau \gg 1$ the previous expansion in τ cannot be used, but in the small region $\tilde{n} \approx 1$ we can find analytically the FCS expanding $\lambda_M(\chi)$ in powers of y. In fact, it turns out that the saddle point equation (12) is solved by $\chi = ix$ with x real and large, thus with $y \ll 1$. We find that the expansion of λ_M contains only even powers of y and at the fourth order coincide with Eq. (30), but with the β coefficients given by (for large τ)

$$\begin{cases} \beta_0 = 2\alpha e^{-4\tau} \\ \beta_1 = 4(1+2\alpha)e^{-4\tau}\tau^2 \\ \beta_2 = \frac{16}{3}(1+2\alpha)e^{-4\tau}\tau^4 \end{cases} \quad (33)$$

For small y it exists a region $\beta_0/\beta_2 < y^2 < \beta_1/\beta_2$ where the $\beta_1 y^2$ term dominates the other two terms. We thus find again the same behavior of Eq. (32) for the probability, but with the coefficients given by Eq. (33). Note that now $\beta_2 > \beta_1$, this means that the probability of transmitting more than one electron per cycle is always larger than the probability of transmitting less than one per cycle. Actually for large a the asymmetry is extreme, the slope for $\tilde{n} < 1$ is much larger than the slope for $\tilde{n} > 1$ which is moderately positive.

For $\tilde{n} \gg 1$ we cannot expand anymore for small y. Since α is not crucial to understand this region we can set $\alpha = 0$ into Eq. (29):

$$\lambda_M(\chi) = e^{-4\tau}\left[y\cosh(2\tau y) + \sinh(2\tau y))\right]^2. \quad (34)$$

For large real y we thus find that the generating function is that of a static grain active [cfr. Eq. (15)] for a fraction $4\phi_0/(2\pi)$ of the time:

$$-S(\chi)/N = 4\phi_0\Gamma\left(e^{i\chi/2}-1\right). \quad (35)$$

IV. GENERAL RESULTS

The results discussed above can be now compared with the numerical results valid for arbitrary values of the amplitude a. These are obtained by solving numerically the system of differential equations (13) to calculate $\hat{A}$. The matrix is then diagonalized and the maximum eigenvalue in modulus selected. Current and noise are obtained by numerical differentiation, while the FCS is obtained by solving numerically Eq. (12).

We begin by discussing current and noise. Fig. 4 shows the average number of electrons and its fluctuation for different values of Γ as a function of the amplitude a. We first notice the qualitative difference between Γ smaller or larger than 1. In the first case the oscillation of the central grain largely increases the current, while in the
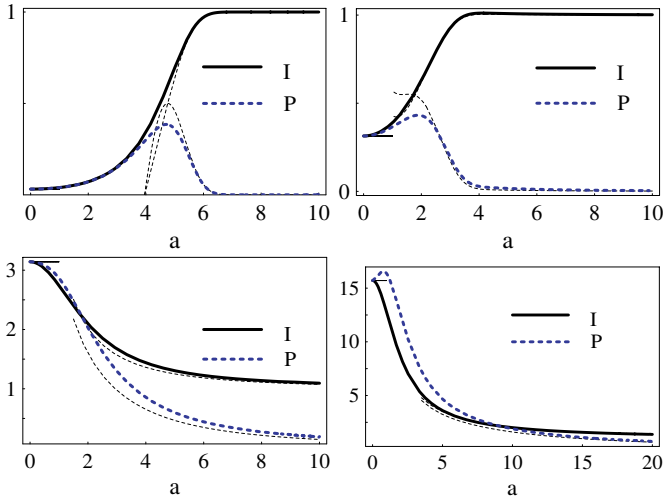
FIG. 4: Current (plain line) and noise (dashed line) as a function of the oscillation amplitude for different values of Γ. From the top-left pannel Γ = 0.01, 0.1, 1, 5. The dashed light lines are obtained with the analytical approximation (29). The short lines at $a = 0$ are the static results given by (15).

FIG. 5: $\ln[P_N(\tilde{n})]/N$ as a function of the number of electrons transferred per cycle $\tilde{n}$ for Γ = 0.1, and oscillation amplitude $a = 0$, 1, 2, 3, 4, and 5 (from left to right).

FIG. 6: The same as Fig. 5 for Γ = 1

second case it reduces it. In both cases for large a the current saturates towards one electron per cycle. From our previous analysis we know that for large Γ the saturation happens only for very large a, when $\tau = 2\Gamma/a$ becomes small enough to reduce the contribution of the central region. (The choice of a factor 2 into the definition of τ is arbitrary and it simply improves the accuracy of the analytical approximation. Any factor of the order of one does not change significantly the results.) A striking feature that appears from the plot for the noise is the enormous reduction of the Fano factor. The transport becomes deterministic due to the shuttling, it is very difficult that the grains perform an oscillation without transmitting one electron.

We believe that measuring noise and current in a device can give a clean indication if the system is actually shuttling electrons. It can discriminate between a simple coupling between the mechanical and the electronic degrees of freedom of the system not associated with the shuttling mechanism.

In Fig. 4 we also plot the comparison with our simple analytical approximation for large and small a. The agreement is pretty good, indicating that the crucial features are correctly reproduced by our simple picture of evolution in five steps.

Let us now discuss the counting statistics. In Fig. 5 we show the evolution of $\ln[P_N(\tilde{n})]/N$ when a is increased from 0 to 5 by steps of one unit. We show the case Γ = 0.1 that is a good representative of the small Γ limit. The full evolution from the static ($a = 0$ and $I = \pi\Gamma$) to the deep shuttling regime ($a = 5$, $I \approx 1$) is obtained (cfr. also the current in Fig. 4). The maximum of the distribution moves from approximately $\pi/10$ to 1. Two features are particularly striking: (i) the peak becomes
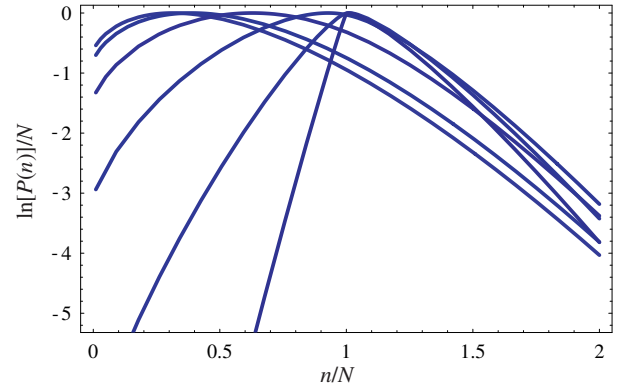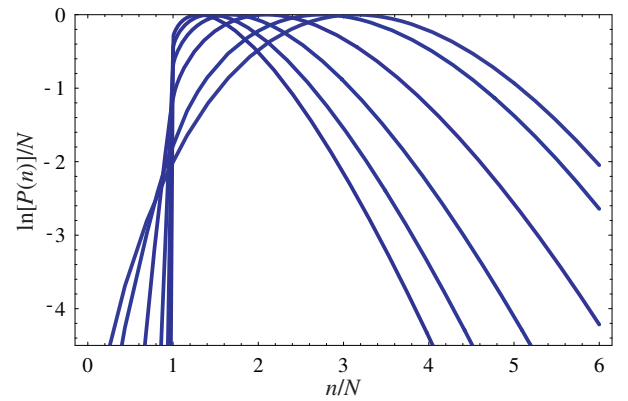
very sharp at the point that a discontinuity of the slope of $\ln P$ appears at $\tilde{n} = 1$ (ii) it becomes asymmetric. The fact that the peak is symmetric in the static case is not surprising, the probability of transferring more or less electrons than the average should not be very different. When a becomes large we have instead shown that those probabilities are controlled by two different parameters, for $\tilde{n} < 1$ by α and for $\tilde{n} > 1$ by $\tau = 2\Gamma/a$. The numerical results confirms this prediction. The behavior around the maximum is well described by the (nearly linear) form (32).

Figure 6 shows the case Γ = 1. In contrast with the previous case now for $a = 0$ the maximum of the distribution if for $\tilde{n} = 2\pi$, larger than 1. Shuttling will reduce the current to 1. The main contribution to the transport comes from the sequential hopping through the grain when both Γ_L and Γ_R are non vanishing. The oscillation reduces this region in favor of regions where only one Γ is non vanishing. In this limit one electron per cycle is transferred. Since this regime is attained when the contribution of the region $x \approx 0$ becomes negligible, $i.e.$ when $\tau = 2\Gamma/a \to 0$, this means that one needs huge oscillation amplitudes to reach the truly shuttling

regime of $\overline{\overline{n}} = 1$. For large a, but not yet in this limit, the probability has the form shown in Fig. 6. We considered also this limit analytically after Eq. (33). Like in the previous cases a singularity develops at 1, but in this case the probability remains monotonic at 1 (for not too large a). The effect of the shuttling is thus mainly to enormously reduce the probability that less than one electron is transferred, and then to slightly shift the maximum in the distribution from $\pi\Gamma > 1$ towards 1. This is due to the fact that due to the oscillations at least one particle is always transferred and the probability of transferring more than one particle is reduced, since the time spent by the shuttle in the central region $|x| \ll \lambda$ is shorter.

V. CONCLUSIONS

In conclusion we have studied the full counting statistics of charge transfer in a single electron transition structure where the central grain can oscillate at a given frequency. The two relevant parameters are the oscillation amplitude divided by the scale of the exponential dependence of the resistance (a), and the probability of a tunnelling event during the time $1/\omega$ for the static structure (Γ). We have obtained both numerical and analytical expressions for the FCS. The results apply to both driven or self oscillating shuttles, when the fluctuation of the amplitude of oscillation can be neglected. The probability of transferring $\tilde{n}$ electrons changes qualitatively as a function of a and Γ. When $\Gamma > 1$ the tunnelling events

happening when the shuttles passes through the region $x \approx 0$ are always important and very large shuttling amplitudes are necessary to have a well defined shuttling regime.

We also discussed in some details the first two moments of the FCS: the current and the noise. We found quantitative prediction for the reduction of the Fano factor for large oscillation amplitudes.

The study of the FCS permits to understand more deeply the dynamics of charge transfer. In some cases we found that the effective elementary charge becomes $1/2$ the actual one, due to correlations. This both in the static and in the dynamic regime. In other limiting cases the statistics is in general polynomial, taking into account the probability of different outcomes at each cycle. Generalization of the theory to a larger number of available states in the grain, or the inclusion of an asymmetric hopping probability is straightforward and can be important to study more realistic systems.

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